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Constructing gradient flows for quasiconvex functionals

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Dedicated to Professor Norio Kikuchi on his sixtieth birthday

1 Introduction

Let $\Omega\subset \boldsymbol{R}^{n}$ be a bounded domain with Lipschitz continuous boundary and let $F=F(x, u, p)$ be a function defined on $\Omega\times \boldsymbol{R}^{N}\times \boldsymbol{R}^{nN}$. First of all we suppose that

$$\min\{n, N\}\leq 2. \tag{1.1}$$

Now let us consider the functional

$$J(u)=\int_{\Omega}F(x, u, Du)dx, \tag{1.2}$$

where $Du=(D_{\alpha}u^{i})=(\frac{\partial u^{i}}{\partial x^{\alpha}})$. The equation of gradient flow for J is given by

$$\frac{\partial u^{i}}{\partial t}(t, x)-\sum_{\alpha=1}^{n}\frac{\partial}{\partial x^{\alpha}}\{F_{p_{\alpha}^{i}}(x, u, Du(x))\}+F_{u^{i}}(x, u, Du(x))=0, \quad x\in\Omega. \tag{1.3}$$

We impose the initial and the boundary conditions

$$u(0, x)=u_{0}(x), \quad x\in\Omega, \tag{1.4}$$

$$u(t, x)=w(x), \quad x\in\Gamma, \tag{1.5}$$

where Γ is a subset of $\partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma)>0$. We suppose that u_{0} and w belong to $W^{1,2}(\Omega, \boldsymbol{R}^{N})$ and that $\gamma u_{0}=\gamma w$ on Γ (γ is the trace operator to $\partial\Omega$).

We say that a function $u\in L^{\infty}((0, \infty);W^{1,2}(\Omega))\cap\bigcup_{T>0}W^{1,2}((0, T)\times\Omega)$ is a *weak solution* to (1.3)–(1.5) if u satisfies $\text{s-}\lim_{t\searrow 0}u(t, x)=u_{0}(x)$ in $L^{2}(\Omega)$, $\gamma u(t)=\gamma w$ on Γ for $\mathcal{L}^{1}$-a.e. t, and for any $\varphi\in C_{0}^{\infty}((0, \infty)\times\Omega)$

$$\begin{aligned}\sum_{i=1}^{N}\int_{0}^{\infty}\int_{\Omega}\{u_{t}^{i}(t, x)\varphi^{i}(t, x)+\sum_{\alpha=1}^{n}F_{p_{\alpha}^{i}}(x, u, Du)D_{\alpha}\varphi^{i}(t, x)\\ +F_{u^{i}}(x, u, Du)\varphi^{i}(t, x)\}dxdt=0.\end{aligned} \tag{1.6}$$

If u is a weak solution to (1.3), then $J(u(t))$ is absolutely continuous and it holds that $dJ(u(t))/dt=-(u_{t}, u_{t})_{L^{2}(\Omega)}\leq 0$ for $\mathcal{L}^{1}$-a.e. t. Thus this defines a gradient flow for J.

We suppose the following facts for the function F.

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(A1) $F \in C^2(\Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN})$

(A2) F is *quasiconvex* with respect to p, that is,

$$\frac{1}{\mathcal{L}^n(D)} \int_D F(x_0, u_0, p_0 + \nabla\varphi(x))dx \geq F(x_0, u_0, p_0)$$

for each bounded domain $D \subset \boldsymbol{R}^n$, for each $(x_0, u_0, p_0) \in \Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$, and for each $\varphi \in W_0^{1,\infty}(D; \boldsymbol{R}^N)$.

(A3) There exist positive constants μ, λ, and a constant γ with $1 \leq \gamma < 2^*$ (the Sobolev exponents for 2) such that

$$\begin{cases} \lambda|p|^2 \leq F(x,u,p) \leq \mu(1 + |u|^\gamma + |p|^2) \\ |F_p|,\ |F_{x,p}|,\ |F_u|,\ |F_{x,u}| \leq \mu(1 + |u|^{\gamma-1} + |p|) \\ |F_{u,u}| \leq \mu(1 + |u|^{\gamma-2}),\ |F_{p,u}|,\ |F_{p,p}| \leq \mu, \end{cases}$$

(A4) There exists a positive constant m such that

$$\sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} \int_\Omega F_{p_\alpha^i p_\beta^j}(x, \psi, D\psi) D_\alpha\varphi^i D_\beta\varphi^j dx \geq m \int_\Omega |D\varphi(x)|^2 dx$$

for any $\psi \in W_0^{1,2}(\Omega, \boldsymbol{R}^N)$, $\varphi \in W_0^{1,2}(\Omega, \boldsymbol{R}^N)$.

Remark. If a quadratic function, $F(p) = \sum a_{ij}^{\alpha\beta} p_\alpha^i p_\beta^j$, satisfies the strong Legendre-Hadamard condition

$$\sum a_{ij}^{\alpha\beta} \xi_\alpha \eta^i \xi_\beta \eta^j \geq \nu|\xi|^2|\eta|^2 \quad (\nu > 0,\ \xi \in \boldsymbol{R}^n,\ \eta \in \boldsymbol{R}^N),$$

then we easily find (A4) holds with $m = \nu$. Thus, if F has the form

$$F(x,u,p) = F_0(p) + G(x,u,p),$$

where F_0 is a quadratic function which satisfies the strong Legendre-Hadamard condition and where G satisfies $|G_{pp}| \leq c\nu$ with $c < 1$, then F satisfies (A4) with $m = (1-c)\nu$.

Example. Let $n = N = 2$. The function

$$\begin{aligned} F(p) = {} & (p_1^1)^2 + (p_2^1)^2 + (p_1^2)^2 + (p_2^2)^2 + 3(p_1^1 p_2^2 - p_2^1 p_1^2) \\ & + \varepsilon\left(\sqrt{1 + (p_1^1)^4 + (p_2^1)^4 + (p_1^2)^4 + (p_2^2)^4 + (p_1^1 p_2^2 - p_2^1 p_1^2)^2} - 1\right) \end{aligned}$$

satisfies (A1)–(A4) if ε is sufficiently small. However this is not convex.

We construct an approximate solution to (1.3)–(1.5) by the method of discretization in time and minimizing variational functionals. In recent several years this approximating way is widely applied to constructing weak solutions to nonlinear partial differential equations.

Let h be a positive number. A sequence $\{u_l\}$ in $W^{1,2}(\Omega, \boldsymbol{R}^N)$ is constructed as follows: we let u_0 be as in (1.4) and for $l \geq 1$ we define u_l as a minimizer of the functional

$$\mathcal{F}_l(v) = \frac{1}{2}\int_\Omega \frac{|v-u_{l-1}|^2}{h}dx + J(v) \qquad (J \text{ is as in } (1.2))$$

in the class $w + W^{1,2}_\Gamma(\Omega, \boldsymbol{R}^N)$, where $W^{1,2}_\Gamma(\Omega, \boldsymbol{R}^N) = \{u \in W^{1,2}(\Omega, \boldsymbol{R}^N); \gamma u = 0 \text{ on } \Gamma\}$ (that is, among functions in $W^{1,2}(\Omega, \boldsymbol{R}^N)$ with $\gamma v = \gamma w$ on Γ). The existence of a minimizer of $\mathcal{F}_l$ is assured by the quasiconvexity of F and (A3) (see, for example, [2, Chapter 4, Theorem 2.9]). Note also that (A3) assures $\mathcal{F}_l$ is Gâteaux differentiable. Approximate solutions $u^h(t,x)$ and $\overline{u}^h(t,x)$ ($(t,x) \in (0,\infty)\times\Omega$) are defined as, for $(l-1)h < t \leq lh$,

$$u^h(t,x) = \frac{t-(l-1)h}{h}u_l(x) + \frac{lh-t}{h}u_{l-1}(x)$$

and

$$\overline{u}^h(t,x) = u_l(x).$$

Then the following facts hold (see, for example, [10] or other references cited in [7]).

Proposition 1.1 *We have*

1) $\{\| u^h_t \|_{L^2((0,\infty)\times\Omega)}\}$ *is uniformly bounded with respect to* h
2) $\{\| \overline{u}^h \|_{L^\infty((0,\infty);W^{1,2}(\Omega))}\}$ *is uniformly bounded with respect to* h
3) $\{\| u^h \|_{L^\infty((0,\infty);W^{1,2}(\Omega))}\}$ *is uniformly bounded with respect to* h
4) *for any* $T>0$, $\{\| u^h \|_{W^{1,2}((0,T)\times\Omega))}\}$ *is uniformly bounded with respect to* h.

Then there exist a function u *such that, passing to a subsequence if necessary,*

5) $\overline{u}^h$ *converges to* u *as* $h \to 0$ *weakly star in* $L^\infty((0,\infty); W^{1,2}(\Omega))$
6) *for any* $T>0$, u^h *converges to* u *as* $h\to 0$ *weakly in* $W^{1,2}((0,T)\times\Omega))$
7) u^h *converges to* u *as* $h \to 0$ *strongly in* $L^2((0,T)\times\Omega)$
8) $\overline{u}^h$ *converges to* u *as* $h \to 0$ *strongly in* $L^2((0,T)\times\Omega)$
9) $s\text{-}\lim_{t\searrow 0} u(t) = u_0$ *in* $L^2(\Omega)$.

Proposition 1.1 9) means that u satisfies (1.4) in a weak sense. Proposition 1.1 5) implies that u satisfies (1.5) in a weak sense since $\overline{u}^h - w \in L^\infty((0,\infty); W^{1,2}_\Gamma(\Omega))$ for each h (note that $W^{1,2}_\Gamma(\Omega)$ is a closed subspace of $W^{1,2}(\Omega)$). Thus the problem is whether u satisfies (1.6). Since u_l is a minimizer of $\mathcal{F}_l(v)$, $d\mathcal{F}_l(u_l+\varepsilon\varphi)/d\varepsilon|_{\varepsilon=0} = 0$ for any $\varphi \in W^{1,2}_0(\Omega)$, and noting that, for $(l-1)h<t<lh$, $u^h_t(t,x) = (u_l(x)-u_{l-1}(x))/h$, we have

$$(1.7)\ \sum_{i=1}^N \int_\Omega \{(u^h_t)^i(x)\varphi^i(x) + \sum_{\alpha=1}^n F_{p^i_\alpha}(x,\overline{u}^h, D\overline{u}^h)D_\alpha\varphi^i(x) + F_{u^i}(x,\overline{u}^h,D\overline{u}^h)\varphi^i(x)\}dx = 0$$

for any $\varphi \in W^{1,2}_0(\Omega)$ and any $t \in \bigcup_{\ell=0}^\infty((\ell-1)h, \ell h)$. This equality leads us to expect that the limit u is a weak solution to (1.3)–(1.5). In fact our main theorem is

Theorem 1.2 (Main Theorem) *u is a weak solution to* (1.3)–(1.5).

It is the same as other nonlinear problems that the difficulty lies in showing the convergence of nonlinear terms. If the functional J is convex, it can be obtained by the use of monotonicity of grad J. But in this article we are assuming only quasiconvexity, and thus it is necessary to introduce a different technique. In [7] a weak solution to a single fourth order parabolic equation is obtained by the use of varifold convergence and Allard's rectifiability theorem, and this method is possibly available for many other equations including our problem. However, since we are treating a vectorial case, the geometrical observations should be more complicated. For example this method requires (1.1) in a geometrical reason.

2 Varifold setting

Let U be an open set of $\boldsymbol{R}^{n+N}$ (we are going to use the varifold theory for the case that $U = \Omega \times \boldsymbol{R}^N$), and let $G = G(n+N, n)$ be the collection of all n-dimensional vector subspaces of $\boldsymbol{R}^{n+N}$. A Radon measure on $U \times G$ is said to be an *n-varifold* in U. Suppose that $\mathcal{M}$ is a countably n-rectifiable set in U (refer to [12, Chapter 3] for the definition and basic properties of an n-rectifiable set) and that θ is a locally $\mathcal{H}^n$-integrable function on $\mathcal{M}$, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. An n-rectifiable varifold $\boldsymbol{v}(\mathcal{M}, \theta)$ is a varifold in U, i.e. a Radon measure on $U \times G$, defined by a continuous linear functional $\varphi \mapsto \int_{\mathcal{M}} \varphi(z, T_z\mathcal{M})\theta(z)d\mathcal{H}^n$, where $T_z\mathcal{M}$ denotes the approximate tangent space of $\mathcal{M}$ at z. We call θ a multiplicity function. When $\theta \equiv 1$, it is simply denoted by $\boldsymbol{v}(\mathcal{M})$. Let V be an n-varifold in U. The weight of V is defined by $\mu_V(B) = V(B \times G)$ for each Borel set $B \subset U$. Clearly μ_V is a Radon measure on U.

Let $I(p, n)$ be the set of multi-indices $\alpha = (\alpha_1, \dots, \alpha_p)$ with $1 \leq \alpha_1 < \cdots < \alpha_p \leq n$. For $\alpha \in I(p, n)$ we set $|\alpha| = p$. $\bar{\alpha} \in I(n-p, n)$ denotes the complement of $\alpha \in I(p, n)$ in $\{1, 2, \dots, n\}$. Let $\{e_1, \dots, e_n, \varepsilon_1, \dots, \varepsilon_N\}$ be the standard orthonormal basis of $\boldsymbol{R}^{n+N}$. We use such notations as $e_\alpha = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_p}$. For convenience we set $I(0, n) = 0$ and $e_0 = \varepsilon_0 = 1$. Let $\bigwedge_n \boldsymbol{R}^{n+N}$ be the set of all n-vectors. For each $\xi \in \bigwedge_n \boldsymbol{R}^{n+N}$ we write

$$\xi = \sum_{|\alpha|+|\beta|=n} \xi^{\alpha\beta} e_\alpha \wedge \varepsilon_\beta. \tag{2.1}$$

Let M be a map from $\boldsymbol{R}^{nN}$ to $\bigwedge_n \boldsymbol{R}^{n+N}$ defined by

$$M(p) = \sum_{|\alpha|+|\beta|=n} \sigma(\alpha, \overline{\alpha}) M^{\beta}_{\overline{\alpha}}(p) e_\alpha \wedge \varepsilon_\beta,$$

where $M^{\beta}_{\overline{\alpha}}$ denotes the $(\overline{\alpha}, \beta)$ minors of p and $M^0_0(p) = 1$. Let Σ_1 denote the image of the map M and Λ_1 be the convex hull of Σ_1 in $\bigwedge_n \boldsymbol{R}^{n+N}$ (see [6, II Section 1.2.1]). These notations are based on [6]. We possibly use other notations without explanations, then consult to [6].

Proposition 2.1 (Theorems 4 and 5 of [6, I Section 3.1.5.]) *Let v be a function in $W^{1,q}(\Omega, \boldsymbol{R}^N)$, $q \geq 1$, and suppose that $|M(Dv(x))| \in L^1(\Omega)$.*

1) *The graph G_v is countably n-rectifiable.*

2) $\mathcal{H}^n(G_v) = \int_\Omega |M(Dv(x))|dx$.

3) *For $\mathcal{L}^n$-a.e. $x \in \Omega$, the approximate tangent space $T_{(x,v(x))}G_v$ exists, and furthermore $|M(Dv(x))|^{-1}M(Dv(x))$ is an n-vector which orients G_v.*

Proposition 2.1 2) implies that for each Borel set $C \subset \Omega$

$$\mathcal{H}^n(\pi^{-1}(C) \cap G_v) = \int_C |M(Dv(x))|dx, \tag{2.2}$$

where π is the projection $U \ni z = (x, y) \mapsto x \in \Omega$. Especially, if $C \subset \Omega$ is an $\mathcal{L}^n$ null set, then $\pi^{-1}(C) \cap G_v$ is an $\mathcal{H}^n$ null set. Hence Proposition 2.1 3) holds for $\mathcal{H}^n$-a.e. $z = (x, v(x)) \in G_v$.

Suppose that $U = \Omega \times \boldsymbol{R}^N$. Let $S \in G(n+N, n)$ and let $\xi = \xi(S)$ be the n-vector which orients S, that is, $\xi(S) = \tau_1 \wedge \cdots \wedge \tau_n$ for an orthonormal basis $\{\tau_1, \cdots, \tau_n\}$ of S. Let $\xi^{\alpha\beta}(S)$ denote the $\alpha\beta$-element of $\xi(S)$ in the expression as in (2.1). For each $S \in G$ there are two orientations. We choose the orientation so that $\xi^{\bar{0}0}(S) \geq 0$. Then the orientation is uniquely determined if $\xi^{\bar{0}0}(S) > 0$. Further we easily obtain that $\xi^{\bar{0}0}$ is continuous on G and $\xi^{\alpha\beta}$ ($|\alpha| + |\beta| = n$, $\beta \neq 0$) is continuous in $G \setminus \mathrm{irr}(G)$, where $\mathrm{irr}(G) = \{S \in G; \xi^{\bar{0}0}(S) = 0\}$. Proposition 2.1 shows $\xi^{\bar{0}0}(T_zG_v) = |M(Dv(\pi(z)))|$ and it is positive for $\mathcal{H}^n$-a.e. $z \in G_v$. Further we have, for $\mathcal{H}^n$-a.e. $z \in G_v$,

$$\xi(T_zG_v) = |M(Dv(\pi(z)))|^{-1}M(Dv(\pi(z))) \tag{2.3}$$

In particular we see that $Dv(x) = ((-1)^{n-i}\xi^{\alpha i}(T_zG_v)/\xi^{\bar{0}0}(T_zG_v))$. By (2.2) we have, for each measurable function g on Ω,

$$\int_\Omega g(x)|M(Dv(x))|dx = \int_{G_v} g(\pi(z))d\mathcal{H}^n(z). \tag{2.4}$$

By the use of (2.4) we have the following proposition.

Proposition 2.2 *Let v be a function in $W^{1,q}(\Omega, \boldsymbol{R}^N)$, $q \geq 1$, and suppose that $|M(Dv(x))| \in L^1_{loc}(\Omega)$. Then there exists a rectifiable varifold $V = \boldsymbol{v}(G_v)$ and satisfies*

$$\int_\Omega f(x, v(x), Dv(x))dx = \int_{U\times G} f(z, \frac{Q_{\xi(S)}}{\xi^{\bar{0}0}(S)})\xi^{\bar{0}0}(S)dV(z, S)$$

for each nonnegative continuous function f on $\Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$, where $Q_\xi = ((-1)^{n-i}\xi^{\alpha i})$.

Proof. By Proposition 2.1 1) the set G_u is countably n-rectifiable, and by 2) we have $\mathcal{H}^n(G_u \cap \Omega' \times \boldsymbol{R}^N) = \int_{\Omega'} |M(Dv(x))|dx < \infty$ for each $\Omega' \subset\subset \Omega$. This shows the function $\theta \equiv 1$ is locally $\mathcal{H}^n$ integrable on G_u. Hence a rectifiable varifold $V = \boldsymbol{v}(G_u, 1) = \boldsymbol{v}(G_u)$ is defined.

When spt f is compact, $f(z, \frac{Q_{\xi(S)}}{\xi^{\bar{0}0}(S)})\xi^{\bar{0}0}(S)$ is continuous in $U \times G$. Thus we have by the definition of n-rectifiable varifold

$$\int_{U\times G} f(z, \frac{Q_{\xi(S)}}{\xi^{\bar{0}0}(S)})\xi^{\bar{0}0}(S)dV(z,S) = \int_{G_v} f(z, \frac{Q_{\xi(T_zG_v)}}{\xi^{\bar{0}0}(T_zG_v)})\xi^{\bar{0}0}(T_zG_v)d\mathcal{H}^n(z)$$

By (2.3) the right hand side of above coincides with $\int_{G_v} |M(Dv(\pi(z))|^{-1}f(z, Dv(\pi(z)))d\mathcal{H}^n$. Applying (2.4) to the case that $g(x) = |M(Dv(x))|^{-1}f(x, v(x), Dv(\pi(z)))$, we obtain the conclusion for a function f with a compact support.

Suppose that f is a general nonnegative continuous function. Then, approximating f with an increasing sequence of functions in $C_0^0(\Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN})$, we obtain the conclusion by the monotone convergence theorem. Q.E.D.

Suppose that $u \in L^\infty((0,\infty); W^{1,2}(\Omega)) \cap \bigcup_{T>0} W^{1,2}((0,T)\times\Omega)$ is a weak solution to (1.3)–(1.5) and satisfies $|M(Du(t,x))| \in L^1_{loc}((0,T)\times\Omega)$. By Proposition 2.1 1) and Proposition 2.2 there is a rectifiable varifold $\boldsymbol{v}(G_{u(t,\cdot)})$ in U for $\mathcal{L}^1$-a.e. t. By (1.6) and Proposition 2.2 we have, for each $\psi(t) \in C_0^\infty(0,\infty)$ and $\phi(z) = \phi(x,y) \in C_0^\infty(U)$ (we use notations x and $z = (x,y)$ for variables in Ω and $U = \Omega \times \boldsymbol{R}^N$, respectively),

$$\text{(2.5)} \quad \sum_{i=1}^N \int_0^\infty \psi(t)\{\int_\Omega u_t^i(t,x)\phi^i(x,u(t,x))dx + \int_{U\times G}[\sum_{\alpha=1}^n F_{p_\alpha^i}(z, \frac{Q_{\xi(S)}}{\xi^{\bar{0}0}(S)})(\frac{\partial\phi^i}{\partial x^\alpha}(z)\xi^{\bar{0}0}(S) + \sum_{j=1}^N \frac{\partial\phi^i}{\partial y^j}(z)(-1)^{n-j}\xi^{\alpha j}(S)) + F_{u^i}(z, \frac{Q_{\xi(S)}}{\xi^{\bar{0}0}(S)})\xi^{\bar{0}0}(S)\phi^i(z)]dV_t(z,S)\}dt = 0,$$

where $V_t = \boldsymbol{v}(G_{u(t,\cdot)})$. Conversely suppose that a function u and a general varifold V_t with a parameter $t \in (0,\infty)$ satisfy (2.5). Then u is a weak solution to (1.3) if

$$\text{(2.6)} \qquad V_t = \boldsymbol{v}(G_{u(t,\cdot)}) \quad \text{for} \quad \mathcal{L}^1\text{-a.e. } t.$$

3 Proof of the Main Theorem (the former half)

Let $u^h(t,x)$ and $\overline{u}^h(t,x)$ be approximate solutions constructed in Section 1. Up to this step we do not require condition (A4). By the use of (A4) we more have the following fact. But this proposition is just a result for the second derivatives with respect to only x variables, and in general we cannot expect the strong convergence of $D\overline{u}^h$. Thus this does not directly imply the convergence of the nonlinear terms.

Proposition 3.1 *For any $\Omega' \subset\subset \Omega$ and for any $T > 0$,*

$$\{\| D_\alpha D_\beta \overline{u}^h \|_{L^2((0,T)\times\Omega')}; \alpha, \beta = 1, 2, \cdots, n\}$$

is uniformly bounded with respect to h.

Proof. The proof is carried out by the use of difference quotient method (refer to the proof of Theorem 1.1 of [5, Chapter II]). In this proof we omit the dependence of t in each functions. Let $\{e_1, \dots, e_n\}$ be the standard basis of $\boldsymbol{R}^n$ and let φ be a function with compact support in Ω. Let k be a sufficiently small positive number depending on the support of φ. We insert $\varphi(x - ke_s)$ in (1.7) instead of $\varphi(x)$, make the change of variables $y = x - ke_s$ in the second and third terms, and rewrite y as x again. Then we have

$$\text{(3.1)}\quad \sum_{i=1}^{N} \int_\Omega \{(u_t^h)^i(x)\varphi^i(x-ke_s) + \sum_{\alpha=1}^{n} F_{p_\alpha^i}(x+ke_s, \overline{u}^h(x+ke_s), D\overline{u}^h(x+ke_s))D_\alpha\varphi^i(x)$$
$$+F_{u^i}(x+ke_s, \overline{u}^h(x+ke_s), D\overline{u}^h(x+ke_s))\varphi^i(x)\}dx = 0.$$

We subtract (1.7) from (3.1) and divide it by k. Then we obtain

$$\text{(3.2)}\qquad -\int_0^1 \sum_{i=1}^{N}\{\int_\Omega (u_t^h)^i(x) D_s\varphi^i(x - \tau ke_s)$$
$$+ \sum_{\alpha=1}^{n}[F_{x^s p_\alpha^i}(\cdots)D_\alpha\varphi^i(x) + \sum_{j=1}^{N} F_{u^j p_\alpha^i}(\cdots)\frac{(\overline{u}^h)^j(x+ke_s) - (\overline{u}^h)^j(x)}{k}D_\alpha\varphi^i(x)$$
$$+ \sum_{j=1}^{N}\sum_{\beta=1}^{n}\int_\Omega F_{p_\alpha^i p_\beta^j}(\cdots)D_\beta(\frac{(\overline{u}^h)^j(x+ke_s) - (\overline{u}^h)^j(x)}{k})D_\alpha\varphi^i(x)]$$
$$+ F_{x^s u^i}(\cdots)\varphi^i(x) + \sum_{j=1}^{N} F_{u^i u^j}(\cdots)\frac{(\overline{u}^h)^j(x+ke_s) - (\overline{u}^h)^j(x)}{k}\varphi^i(x)$$
$$+ \sum_{j=1}^{N}\sum_{\beta=1}^{n}\int_\Omega F_{u^i p_\beta^j}(\cdots)D_\beta(\frac{(\overline{u}^h)^j(x+ke_s) - (\overline{u}^h)^j(x)}{k})\varphi^i(x)\}dxd\tau = 0,$$

where

$$(\cdots) = (x + \tau ke_s, \tau\overline{u}^h(x + ke_s) + (1-\tau)\overline{u}^h(x), \tau D\overline{u}^h(x+ke_s) + (1-\tau)D\overline{u}^h(x)).$$

Let B_R be a ball with radius R. We take $\eta \in C_0^\infty(B_R)$ such that $B_{3R} \subset\subset \Omega$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{R/2}$, and $|D\eta| \le C/R$. Now we insert in (3.2)

$$\varphi(x) = \frac{\overline{u}^h(x+ke_s) - \overline{u}^h(x)}{k}\eta(x)^2.$$

Note that, by the change of variables $x \mapsto x - \tau ke_s$, we have

$$I := \sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{n}\int_\Omega F_{p_\alpha^i p_\beta^j}(\cdots)$$
$$D_\beta(\frac{(\overline{u}^h)^j(x+ke_s) - (\overline{u}^h)^j(x)}{k}\eta(x))D_\alpha(\frac{(\overline{u}^h)^i(x+ke_s) - (\overline{u}^h)^i(x)}{k}\eta(x))dx$$
$$= \sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{n}\int_\Omega F_{p_\alpha^i p_\beta^j}(\cdots)D_\beta(\frac{(\overline{u}^h)^j(x-\tau ke_s+ke_s) - (\overline{u}^h)^j(x-\tau ke_s)}{k}\eta(x-\tau ke_s))$$
$$D_\alpha(\frac{(\overline{u}^h)^i(x-\tau ke_s+ke_s) - (\overline{u}^h)^i(x-\tau ke_s)}{k}\eta(x-\tau ke_s))dx,$$

where $(\cdots)$ has been changed to

$$(x, \tau\overline{u}^h(x-\tau ke_s+ke_s)+(1-\tau)\overline{u}^h(x-\tau ke_s), \tau D\overline{u}^h(x-\tau ke_s+ke_s)+(1-\tau)D\overline{u}^h(x-\tau ke_s)).$$

Then by the use of (A4)

$$I \geq m\int_\Omega |D(\frac{\overline{u}^h(x-\tau ke_s+ke_s)-\overline{u}^h(x-\tau ke_s)}{k}\eta(x-\tau ke_s))|^2dx.$$

Returning the variables $x-\tau ke_s \mapsto x$, we obtain by (3.2) and (A3)

$$\int_{B_{R/2}}|D(\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k})|^2dx \leq C(R,\eta)\{\int_\Omega |u_t^h(x)|^2dx+\int_{B_R}|\frac{\overline{u}^h(x+ke_s)-\overline{u}^h(x)}{k}|^2dx\}.$$

This implies for any $T>0$

$$\int_0^T\int_{B_{R/2}}|D_sD\overline{u}^h(x)|^2dxdt \leq C(R,\eta)\{\int_0^T\int_\Omega |u_t^h(x)|^2dxdt+\int_0^T\int_{B_{2R}}|D\overline{u}^h(x)|^2dxdt\}.$$

It follows from Proposition 1.1 2), 4) that the right hand side of the above inequality is less than a constant which is independent of h. This completes the proof. Q.E.D.

In this article p' and p^* denote the dual and the Sobolev exponents for p, respectively.

Corollary 3.2 *For any $\Omega' \subset\subset \Omega$ and for any $T>0$ we have*
1) $\{\| D\overline{u}^h \|_{L^{s_0}((0,T)\times\Omega')}\}$ is uniformly bounded with respect to h, where $s_0 = 2/(2^*/2)'+2 = 4/n+2$ if $n\geq 3$, $=4$ if $n=1, 2$
2) $\{\| \overline{u}^h \|_{L^{2^*}((0,T)\times\Omega')}\}$ is uniformly bounded with respect to h.

Proof. By Hölder's inequality

$$\begin{aligned}\int_0^T\int_{\Omega'}|D\overline{u}^h|^{s_0}dxdt &= \int_0^T\int_{\Omega'}|D\overline{u}^h|^2|D\overline{u}^h|^{s_0-2}dxdt \\ &\leq \int_0^T(\int_{\Omega'}|D\overline{u}^h|^{2^*}dx)^{2/2^*}(\int_{\Omega'}|D\overline{u}^h|^{(s_0-2)(2^*/2)'}dx)^{1/(2^*/2)'}dt.\end{aligned}$$

Since s_0 is defined so that $(s_0-2)(2^*/2)'=2$, we have

$$\int_0^T\int_{\Omega'}|D\overline{u}^h|^{s_0}dxdt \leq \| D\overline{u}^h \|^{2/(2^*/2)'}_{L^\infty((0,T);L^2(\Omega'))}\| D\overline{u}^h \|^2_{L^2((0,T);L^{2^*}(\Omega'))}. \tag{3.3}$$

It follows from Sobolev's imbedding theorem and Proposition 3.1 that $\| D\overline{u}^h \|_{L^2((0,T);L^{2^*}(\Omega'))}$ is uniformly bounded with respect to h. Hence the right hand side of (3.3) is uniformly bounded by Proposition 1.1 2).

Assertion 2) immediately follows from Assertion 1) and Sobolev's imbedding theorem (note that we can obtain higher integrability than 2^*). Q.E.D.

Proposition 3.3 *For any $\Omega' \subset\subset \Omega$ and for any $T>0$ there exists $q_0>1$ such that $\{\| M(D\overline{u}^h) \|_{L^{q_0}((0,T)\times\Omega')}\}$ is uniformly bounded with respect to h.*

Proof. By (1.1) we have $s_0 > \min\{n, N\}$. Hence the conclusion immediately follows from Corollary 3.2. Q.E.D.

Corollary 3.2 and Proposition 3.3 imply that there exists a constant $K_0 = K_0(T, \Omega')$ which is independent of h such that

$$\| D\overline{u}^h \|_{L^{s_0}((0,T)\times\Omega')}, \ \| \overline{u}^h \|_{L^{2^*}((0,T)\times\Omega')}, \ \| M(D\overline{u}^h) \|_{L^{q_0}((0,T)\times\Omega')} \le K_0. \tag{3.4}$$

In particular we have, for $\mathcal{L}^1$-a.e. t, $|M(D\overline{u}^h(t,\cdot))| \in L^{q_0}_{loc}(\Omega)$. Thus by Proposition 2.2 there exists a one parameter family of rectifiable varifolds

$$V_t^h = \boldsymbol{v}(G_{\overline{u}^h(t,\cdot)}).$$

Further by Proposition 2.2 again we can rewrite (1.7) as (2.5): for each $\psi(t) \in C_0^\infty(0,\infty)$ and $\phi(z) \in C_0^\infty(U)$

$$\begin{aligned}
\sum_{i=1}^{N} \int_0^\infty \psi(t)\{ &\int_\Omega (u_t^h)^i(t,x)\phi^i(x, \overline{u}^h(t,x))dx \\
+ &\int_{U\times G}[\sum_{\alpha=1}^{n} F_{p^i_\alpha}(z, \frac{Q_{\xi(S)}}{\xi^{\bar{0}0}(S)})(\frac{\partial\phi^i}{\partial x^\alpha}(z)\xi^{\bar{0}0}(S) + \sum_{j=1}^{N}\frac{\partial\phi^i}{\partial y^j}(z)(-1)^{n-j}\xi^{\alpha j}(S)) \\
+ &F_{u^i}(z, \frac{Q_{\xi(S)}}{\xi^{\bar{0}0}(S)})\xi^{\bar{0}0}(S)\phi^i(z)]dV_t^h(z,S)\}dt = 0.
\end{aligned} \tag{3.5}$$

By (3.4) and Hölder's inequality we have $(\int_0^T (\int_{\Omega'} |M(D\overline{u}^h(t,x))|dx)^{q_0}dt)^{1/q_0} \le K$, where $K = K(T,\Omega') = \mathcal{L}^n(\Omega')^{1/q_0'}K_0(T,\Omega')$. Noting that $\int_{U'\times G} dV_t^h = \int_{U'} \mu_{V_t^h} = \mathcal{H}^n(G_{\overline{u}^h(t,\cdot)} \cap U')$, where $U' = \Omega' \times \boldsymbol{R}^N$, we have by Proposition 2.1 2)

$$(\int_0^T | \int_{U\times G} \varphi(z,S)dV_t^h(z,S)|^{q_0}dt)^{1/q_0} \le K \sup|\varphi| \tag{3.6}$$

for any $\varphi(z,S) \in C_0^0(U\times G)$ with spt $\varphi \subset \Omega' \times \boldsymbol{R}^N \times G$ and $K = K(T,\Omega')$.

The following proposition can be obtained by the use of (3.6) in the same way as in the proof of Proposition 4.3 of [4].

Proposition 3.4 *There exists a subsequence of* $\{V_t^h\}$ *(still denoted by* $\{V_t^h\}$*) and a one parameter family of varifolds* V_t *in* U*, for* $t \in (0,\infty)$*, such that, for each* $\psi(t) \in L^{q_0'}(0,T)$ *and* $\varphi(z,S) \in C_0^0(U\times G)$,

$$\lim_{h\to 0} \int_0^T \psi(t) \int_{U\times G} \varphi(z,S)dV_t^h(z,S)dt = \int_0^T \psi(t) \int_{U\times G} \varphi(z,S)dV_t(z,S)dt.$$

Lemma 3.5 *Let* q_0 *and* γ *be as in Proposition* 3.3 *and* (A3)*, respectively, and suppose that* $1 < q < q_0$*. Then for any* $\Omega' \subset\subset \Omega$, $\{1 + |\overline{u}^h(t,x)|^\gamma + |D\overline{u}^h(t,x)|^2 + |M(D\overline{u}^h(t,x))|^q\}$ *is equiintegrable in* $(0,T)\times\Omega'$ *with respect to* h.

Proof. Let $s_1 = (2^*/\gamma)'$, $s_2 = (s_0/2)'$, $s_3 = (q_0/q)'$. Since $\gamma < 2^*$, $2 < s_0$, and $q < q_0$, we have $1 \le s_1, s_2, s_3 < \infty$. By Hölder's inequality and Corollary 3.2 we have for each subset $E \subset (0,T) \times \Omega'$

$$\begin{aligned}
&\iint_E (1 + |\overline{u}^h(t,x)|^\gamma + |D\overline{u}^h(t,x)|^2 + |M(D\overline{u}^h(t,x))|^q) dxdt \\
\le\ & \mathcal{L}^{n+1}(E) + (\iint_E 1^{s_1} dxdt)^{1/s_1} \| \overline{u}^h \|^\gamma_{L^{2^*}((0,T)\times\Omega')} \\
&+ (\iint_E 1^{s_2} dxdt)^{1/s_2} \| D\overline{u}^h \|^2_{L^{s_0}((0,T)\times\Omega')} + (\iint_E 1^{s_3} dxdt)^{1/s_3} \| M(D\overline{u}^h) \|^q_{L^{q_0}((0,T)\times\Omega')} \\
\le\ & \mathcal{L}^{n+1}(E) + \mathcal{L}^{n+1}(E)^{1/s_1} K_0^\gamma + \mathcal{L}^{n+1}(E)^{1/s_2} K_0^2 + \mathcal{L}^{n+1}(E)^{1/s_3} K_0^q,
\end{aligned}$$

which implies the conclusion. Q.E.D.

Proposition 3.6 *Let $f(z,p)$ be a continuous function on $U \times \boldsymbol{R}^n$ and let q be as in Lemma 3.5. Suppose that $\{x; f(x,y,p) \neq 0$ for some $(y,p)\} \subset \Omega'$ for a set $\Omega' \subset\subset \Omega$ and that for each $z = (x,y) \in U$ and each $p \in \boldsymbol{R}^{nN}$*

$$|f(z,p)| \le \mu_1(1 + |y|^\gamma + |p|^2 + |M(p)|^q) \tag{3.7}$$

holds with a constant μ_1. Let T be any positive number. Then, if $\{V_t^h\}$ and V_t are as in Proposition 3.4, for each $\psi(t) \in L^\infty(0,T)$ we have

$$\lim_{h\to 0} \int_0^T \psi(t) \int_{U\times G} f(z, \frac{Q_{\xi(S)}}{\xi^{\bar{0}0}(S)}) \xi^{\bar{0}0}(S) dV_t^h(z,S) dt = \int_0^T \psi(t) \int_{U\times G} f(z, \frac{Q_{\xi(S)}}{\xi^{\bar{0}0}(S)}) \xi^{\bar{0}0}(S) dV_t(z,S) dt.$$

Proof. Let ι be an increasing continuous function on $\boldsymbol{R}$ with $\iota(r) = 1$ for $r > 2$, $\iota = 0$ for $r < 1$, and we put $\xi_\varepsilon^{\bar{0}0}(S) = \xi^{\bar{0}0}(S)\iota(\xi^{\bar{0}0}(S)/\varepsilon)$ and $f_\varepsilon(z,p) = f(z,p)(1 - \iota(\varepsilon e(y,p)))$, where $e(y,p) = 1 + |y|^\gamma + |p|^2 + M(p)^q$. Note that $\xi_\varepsilon^{\bar{0}0}(S) = \xi^{\bar{0}0}(S)$ for $\xi^{\bar{0}0}(S) \ge 2\varepsilon$, $f_\varepsilon = f$ for $e(y,p) \le \varepsilon^{-1}$, $f_\varepsilon = 0$ for $e(y,p) \ge 2\varepsilon^{-1}$. Thus $f_\varepsilon(z,p)\xi_\varepsilon^{\bar{0}0}(S)$ converges to $f(z,p)\xi^{\bar{0}0}(S)$ for each $(z,S) \in U \times G$. Further we have $|f_\varepsilon(z,p)| \le 2\mu_1\varepsilon^{-1}$ by (3.7). For simplicity, we omit the dependence on S of Q_ξ, $\xi^{\bar{0}0}$, and $\xi_\varepsilon^{\bar{0}0}$. Now

$$\begin{aligned}
&| \int_0^T \psi(t) \int_{U\times G} f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) \xi_\varepsilon^{\bar{0}0} dV_t^h(z,S) dt - \int_0^T \psi(t) \int_{U\times G} f(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) \xi^{\bar{0}0} dV_t^h(z,S) dt | \\
\le\ & \int_0^T |\psi(t)| \int_{U\times G} |f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}})| |\xi_\varepsilon^{\bar{0}0} - \xi^{\bar{0}0}| dV_t^h(z,S) dt \\
&+ \int_0^T |\psi(t)| \int_{U\times G} |f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) - f(z, \frac{Q_\xi}{\xi^{\bar{0}0}})| \xi^{\bar{0}0} dV_t^h(z,S) dt.
\end{aligned} \tag{3.8}$$

It follows from Proposition 2.2 that

$$\begin{aligned}
&\int_0^T |\psi(t)| \int_{U\times G} |f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}})| |\xi_\varepsilon^{\bar{0}0} - \xi^{\bar{0}0}| dV_t^h(z,S) dt \\
=\ & \int_0^T |\psi(t)| \int_{U\times G} |f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}})| |1 - \iota(\xi^{\bar{0}0}/\varepsilon)| \xi^{\bar{0}0} dV_t^h(z,S) dt
\end{aligned} \tag{3.9}$$

$$\leq \int_0^T |\psi(t)| \int_{\Omega'} |f_\varepsilon(x, \overline{u}^h(t,x), D\overline{u}^h(t,x))||1 - \iota(1/\varepsilon |M(D\overline{u}^h(t,x))|)| dxdt$$
$$\leq \iint_{E_\varepsilon} |\psi(t)||f_\varepsilon(x, \overline{u}^h(t,x), D\overline{u}^h(t,x))| dxdt$$
$$\leq \mu_1 \, \|\psi\|_{L^\infty(0,T)} \iint_{E_\varepsilon} (1 + |\overline{u}^h(t,x)|^\gamma + |D\overline{u}^h(t,x)|^2 + |M(D\overline{u}^h(t,x))|^q) dxdt,$$

where $E_\varepsilon = \{(t,x); |M(D\overline{u}^h(t,x))| \geq (2\varepsilon)^{-1}\} \cap (0,T) \times \Omega'$. It also follows from Proposition 2.2 that

$$(3.10) \qquad \int_0^T |\psi(t)| \int_{U\times G} |f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) - f(z, \frac{Q_\xi}{\xi^{\bar{0}0}})| \xi^{\bar{0}0} dV_t^h(z,S) dt$$
$$= \int_0^T |\psi(t)| \int_{U\times G} |f(z, \frac{Q_\xi}{\xi^{\bar{0}0}})||\iota(\varepsilon e(y, -\frac{Q_\xi}{\xi^{\bar{0}0}}))| \xi^{\bar{0}0} dV_t^h(z,S) dt$$
$$\leq \int_0^T |\psi(t)| \int_{\Omega'} |f(x, \overline{u}^h(t,x), D\overline{u}^h(t,x))||\iota(\varepsilon e(\overline{u}^h(t,x), D\overline{u}^h(t,x)))| dxdt$$
$$\leq \iint_{F_\varepsilon} |\psi(t)||f(x, \overline{u}^h(t,x), D\overline{u}^h(t,x))| dxdt$$
$$\leq \mu_1 \, \|\psi\|_{L^\infty(0,T)} \iint_{F_\varepsilon} (1 + |\overline{u}^h(t,x)|^\gamma + |D\overline{u}^h(t,x)|^2 + |M(D\overline{u}^h(t,x))|^q) dxdt,$$

where $F_\varepsilon = \{(t,x); e(\overline{u}^h(t,x), D\overline{u}^h(t,x)) \geq \varepsilon^{-1}\} \cap (0,T) \times \Omega'$. By Chebyshev's inequality we see that $\mathcal{L}^{n+1}(E_\varepsilon) < (2K_0\varepsilon)^{q_0}$ and $\mathcal{L}^{n+1}(F_\varepsilon) < K_1\varepsilon$, where K_0 is as in (3.4) and

$$(3.11) \quad K_1 = T\mathcal{L}^n(\Omega') + (T\mathcal{L}^n(\Omega'))^{1/s_1} K_0^\gamma + (T\mathcal{L}^n(\Omega'))^{1/s_2} K_0^2 + (T\mathcal{L}^n(\Omega'))^{1/s_3} K_0^q$$

with s_1, s_2, s_3 as in the proof of Lemma 3.5. By (3.8), (3.9), (3.10), and Lemma 3.5 we have that, as $\varepsilon \to 0$,

$$\int_0^T \psi(t) \int_{U\times G} f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) \xi_\varepsilon^{\bar{0}0} dV_t^h(z,S) dt \longrightarrow \int_0^T \psi(t) \int_{U\times G} f(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) \xi^{\bar{0}0} dV_t^h(z,S) dt$$

uniformly with respect to h.

On the other hand, since $f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) \xi_\varepsilon^{\bar{0}0} \in C_0^0(U \times G)$, Proposition 3.4 implies

$$(3.12) \; \lim_{h\to 0} \int_0^T \psi(t) \int_{U\times G} f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) \xi_\varepsilon^{\bar{0}0} dV_t^h(z,S) dt = \int_0^T \psi(t) \int_{U\times G} f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) \xi_\varepsilon^{\bar{0}0} dV_t(z,S) dt.$$

By Proposition 2.2, (3.4), and (3.7) we have

$$|\int_0^T \psi(t) \int_{U\times G} f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) \xi_\varepsilon^{\bar{0}0} dV_t^h(z,S) dt| \leq \int_0^T |\psi(t)| \int_{U\times G} |f(z, \frac{Q_\xi}{\xi^{\bar{0}0}})| \xi^{\bar{0}0} dV_t^h(z,S) dt$$
$$\leq \mu_1 \int_0^T |\psi(t)| \int_{\Omega'} (1 + |\overline{u}^h(t,x)|^\gamma + |D\overline{u}^h(t,x)|^2 + |M(D\overline{u}^h(t,x))|^q) dxdt \leq K_1\mu_1 \, \|\psi\|_{L^\infty(0,T)},$$

where K_1 is as in (3.11). This and (3.12) imply

$$|\int_0^T \psi(t) \int_{U\times G} f_\varepsilon(z, \frac{Q_\xi}{\xi^{\bar{0}0}}) \xi_\varepsilon^{\bar{0}0} dV_t(z,S) dt| \leq K_1\mu_1 \, \|\psi\|_{L^\infty(0,T)} \, .$$

Without loss of generality we may assume that f and ψ are nonnegative, and then we have $\psi(t)f_\varepsilon(z,p)\xi_\varepsilon^{\bar{0}0} < \psi(t)f_{\varepsilon'}(z,p)\xi_{\varepsilon'}^{\bar{0}0}$ whenever $\varepsilon > \varepsilon'$. Hence by the monotone convergence theorem we have, as $\varepsilon \to 0$,

$$\int_0^T \psi(t)\int_{U\times G} f_\varepsilon(z,\frac{Q_\xi}{\xi^{\bar{0}0}})\xi_\varepsilon^{\bar{0}0}dV_t(z,S)dt \longrightarrow \int_0^T \psi(t)\int_{U\times G} f(z,\frac{Q_\xi}{\xi^{\bar{0}0}})\xi^{\bar{0}0}dV_t(z,S)dt.$$

Thus we have the conclusion by the use of a standard fact in iterated limits. Q.E.D.

Proposition 3.7 *The function u of Proposition* 1.1 *and V_t of Proposition* 3.4 *satisfy* (2.5).

Proof. Possibly passing to further subsequences, we have by Proposition 1.1 5) and 6) that the first integral of (3.5) converges to that of (2.5) as $h \to 0$. Since, in (3.5), spt $\psi \subset (0,T)$ for some T, we apply Proposition 3.6 to the case that

$$f(z,p) = \sum_{i=1}^N [\sum_{\alpha=1}^n F_{p_\alpha^i}(z,p)(\frac{\partial\phi^i}{\partial x^\alpha}(z) + \sum_{j=1}^N \frac{\partial\phi^i}{\partial y^j}(z)p_\alpha^j) + F_{y^i}(z,p)\phi^i(z)].$$

Then the second integral of (3.5) converges to that of (2.5) as $h \to 0$. Q.E.D.

4 Proof of the Main Theorem (the latter half)

Proposition 3.7 implies that Theorem 1.2 is equivalent to (2.6). There are three steps in proving (2.6):

Step 1. *μ_{V_t} and $\mathcal{H}^n \llcorner G_{u(t,\cdot)}$ are mutually absolutely continuous for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$.*

(This step implies in particular that spt $\mu_{V_t} =$ spt $\mathcal{H}^n \llcorner G_{u(t,\cdot)}$.)

Step 2. *V_t is an n-rectifiable varifold $\boldsymbol{v}(G_{u(t,\cdot)},\theta_t)$ for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$.*

Step 3. *$\theta_t(z) = 1$ for $\mathcal{H}^n$-a.e. $z \in G_{u(t,\cdot)}$, for $\mathcal{L}^1$-a.e. t.*

Lemma 4.1 *Put $\mathcal{C} = \{(f_{\alpha\beta})_{|\alpha|+|\beta|=n} \in C_0^0(\Omega)^{p(n+N,n)};\ \sqrt{\sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x)^2} \le 1$ for $x \in \Omega\}$, where $p(n+N,n) = (n+N)!/n!N!$. For each $\psi \in L^\infty(0,T)$, $\phi \in C_0^0(U)$, and $(f_{\alpha\beta}) \in \mathcal{C}$ we have*

$$\int_0^T \psi(t)\int_\Omega \phi(x,u(t,x)) \sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x)M_{\bar{\alpha}}^{\beta}(Du(t,x))dxdt$$
$$= \int_0^T \psi(t)\int_{U\times G} \phi(z) \sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x)\xi^{\alpha\beta}(S)dV_t(z,S)dt.$$

Proof. It follows from Proposition 1.1 5), 8), Proposition 3.1, and the Piola identities that $M_{\bar{\alpha}}^{\beta}(D\overline{u}^h(t,x))$ converges to $M_{\bar{\alpha}}^{\beta}(D\overline{u}^h(t,x))$ weakly in $L^{q_0}((0,T)\times\Omega')$. Thus, for any $\psi \in L^\infty(0,T)$,

$$(4.1)\quad \lim_{h\to 0}\int_0^T \psi(t)\int_\Omega \phi(x,\overline{u}^h(t,x))\sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x)M_{\overline{\alpha}}^{\beta}(D\overline{u}^h(t,x))dxdt$$
$$=\int_0^T \psi(t)\int_\Omega \phi(x,u(t,x))\sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x)M_{\overline{\alpha}}^{\beta}(Du(t,x))dxdt.$$

On the other hand, since Proposition 2.2 implies

$$\int_\Omega \phi(x,\overline{u}^h(t,x))\sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x)M_{\overline{\alpha}}^{\beta}(D\overline{u}^h(t,x))dx$$
$$=\int_{U\times G}\phi(z)\Big(\sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x)\frac{\xi^{\alpha\beta}(S)}{\xi^{\overline{0}0}(S)}\Big)\xi^{\overline{0}0}(S)dV_t^h(z,S),$$

we have by Proposition 3.6

$$(4.2)\quad \lim_{h\to 0}\int_0^T \psi(t)\int_\Omega \phi(x,\overline{u}^h(t,x))\sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x)M_{\overline{\alpha}}^{\beta}(D\overline{u}^h(t,x))dxdt$$
$$=\int_0^T \psi(t)\int_{U\times G}\phi(z)\Big(\sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x)\frac{\xi^{\alpha\beta}(S)}{\xi^{\overline{0}0}(S)}\Big)\xi^{\overline{0}0}(S)dV_t(z,S)dt$$

Thus the conclusion follows from (4.1) and (4.2). Q.E.D.

When $f_{\overline{0}0}\equiv 1$ on $\pi(\mathrm{spt}\ \phi)$ and $f_{\alpha\beta}\equiv 0$ ($|\beta|\neq 0$) in Lemma 4.1, we have

$$(4.3)\quad \int_0^T \psi(t)\int_\Omega \phi(x,u(t,x))dxdt=\int_0^T \psi(t)\int_{U\times G}\phi(z)\xi^{\overline{0}0}(S)dV_t(z,S)dt.$$

Proof of Step 1. By (2.4) we have

$$\int_\Omega \phi(x,u(t,x))dx=\int_U \phi(z)|M(Du(\pi(z)))|^{-1}d(\mathcal{H}^n\llcorner G_{u(t,\cdot)}).$$

By Lemma 38.4 of [12] there is a probability Radon measure $\eta_{V_t}^{(z)}$ for μ_{V_t}-a.e. $z\in U$ such that

$$\int_{U\times G}\beta(z,S)dV_t(z,S)=\int_U\Big(\int_G \beta(z,S)d\eta_{V_t}^{(z)}(S)\Big)d\mu_{V_t}.$$

Then we have by (4.3) that, for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$,

$$\int_U \phi(z)|M(Du(\pi(z)))|^{-1}d(\mathcal{H}^n\llcorner G_{u(t,\cdot)})=\int_U \phi(z)\Big(\int_G \xi^{\overline{0}0}(S)d\eta_{V_t}^{(z)}(S)\Big)d\mu_{V_t}$$

for each $\phi\in C_0^0(U)$. This means, for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$,

$$(4.4)\quad \int_A |M(Du(\pi(z)))|^{-1}d(\mathcal{H}^n\llcorner G_{u(t,\cdot)})=\int_A\Big(\int_G \xi^{\overline{0}0}(S)d\eta_{V_t}^{(z)}(S)\Big)d\mu_{V_t}$$

for each Borel set $A\subset U$.

When a Borel set A satisfies $\mu_{V_t}(A)=0$, we have $\int_A |M(Du(\pi(z)))|^{-1}d(\mathcal{H}^n\llcorner G_{u(t,\cdot)})=0$ by (4.4), which implies $(\mathcal{H}^n\llcorner G_{u(t,\cdot)})(A)=0$. Thus $\mathcal{H}^n\llcorner G_{u(t,\cdot)}$ is absolutely continuous with respect to μ_{V_t} for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$.

Conversely, when $(\mathcal{H}^n\llcorner G_{u(t,\cdot)})(A)=0$, we have $\int_{A\times G}\xi^{\overline{0}0}(S)dV_t(z,S)=0$ by (4.4). Then $V_t(A\times(G\setminus \mathrm{irr}(G)))=0$ for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$. On the other hand, noting

$\xi^{\bar{0}0} = |M(D(\overline{u}^h(t,x))|^{-1}$ and $M(D\overline{u}^h(t,x)) \in L^{q_0}_{loc}([0,\infty)\times\Omega)$, we have by Proposition 3.6 that, for any positive number T, for each $\psi \in L^\infty(0,T)$ and $\phi \in C^0_0(U)$, and $1 < q < q_0$,

$$\lim_{h\to 0}\int_0^T \psi(t)\int_\Omega \phi(x,\overline{u}^h(t,x))|M(D\overline{u}^h(t,x))|^q dxdt = \int_0^T \psi(t)\int_{U\times G}\phi(z)\xi^{\bar{0}0}(S)^{-(q-1)}dV_t(z,S)dt.$$

By (3.4) the left hand side of the above is less than $K_0 \parallel \psi \parallel_{L^\infty(0,T)} \sup|\phi|$. This implies $V_t(U\times \operatorname{irr}(G)) = 0$ for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$. Thus we have $\mu_{V_t}(A) = V_t(A\times G) = 0$ for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$. Then μ_{V_t} is absolutely continuous with respect to $\mathcal{H}^n \llcorner G_{u(t,\cdot)}$ for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$. Q.E.D.

In our theory we need Allard's rectifiability theorem ([1]). We define for $S\in G(n+N,n)$ and $X = (X^1,\dots,X^{n+N}) \in C^1_0(U;\boldsymbol{R}^{n+N})$

$$\operatorname{div}_S X = \sum_{i,j=1}^{n+N} p_S^{ij}\frac{\partial X^i}{\partial x^j}. \tag{4.5}$$

The first variation δV for an n-varifold V on U is given by

$$\delta V(X) = \int_{U\times G}\operatorname{div}_S X(x)dV(x,S). \tag{4.6}$$

We say that V has locally bounded first variation in U if for each $W \subset\subset U$ and each $X\in C^1_0(U;\boldsymbol{R}^{n+N})$ with $\operatorname{spt}X \subset W$ there exists a constant $C>0$ such that $|\delta V(X)| \le C\sup|X|$. Let $\Theta^{*n}(\mu_V,z)$, $\Theta^n_*(\mu_V,z)$ denote the upper and lower densities of μ_V at $z\in U$, respectively. If $\Theta^{*n}(\mu_V,z) = \Theta^n_*(\mu_V,z)$, this common value is denoted by $\Theta^n(\mu_V,z)$ and it is called the n-dimensional density of μ_V at z. Now we assert Allard's Rectifiability Theorem (refer to, for example, [12, Theorem 42.4]).

Allard's Rectifiability Theorem *Suppose that V has locally bounded first variation in U and $\Theta^n(\mu_V,z) > 0$ for μ_V-a.e. $z\in U$. Then V is n-rectifiable.*

Allard's Rectifiability Theorem implies that Step 2 immediately follows if we obtain the following two facts.

Lemma 4.2 *V_t has locally finite first variation for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$.*

Lemma 4.3 *$\Theta^n(\mu_{V_t},z)\ge 1$ for μ_{V_t}-a.e. $z\in U$, for $\mathcal{L}^1$-a.e. $t\in(0,\infty)$.*

Proof of Lemma 4.2. Let T be any positive number and let W be any open set in U such that $\Omega' := \pi(W) \subset\subset \Omega$. Suppose that $X\in C^1_0(U;\boldsymbol{R}^{n+N})$ satisfies $\operatorname{spt} X\subset W$. Note that by (4.6)

$$(\delta V_t^h)(X) = \int_{U\times G}\operatorname{div}_S X dV_t^h = \int_{G_u}\operatorname{div}_M X d\mathcal{H}^n,$$

where $M = T_{(x,\overline{u}^h(t,x))}G_{\overline{u}^h(t,x)}$. Then we have by (4.5)

$$(\delta V_t^h)(X) = \int_{G_u}\sum_{i,j=1}^{n+N} p_M^{ij}\frac{\partial X^i}{\partial z^j}d\mathcal{H}^n = \int_\Omega \sum_{i,j=1}^{n+N} p_M^{ij}\frac{\partial X^i}{\partial z^j}|M(D\overline{u}^h)|dx.$$

Since $\dfrac{\partial X^i}{\partial z^j} = \dfrac{\partial}{\partial x^j}(X^i(x,\overline{u}^h) - \sum_{k=1}^{N} \dfrac{\partial X^i}{\partial y^k}(x,\overline{u}^h)\dfrac{\partial(\overline{u}^h)^k}{\partial x^j})$ $(j=1,2,\dots,n)$, it holds that

$$(4.7)\sum_{i,j=1}^{n+N} p_M^{ij}\frac{\partial X^i}{\partial z^j} = \sum_{i=1}^{n+N}[\sum_{j=1}^{n} p_M^{ij}\frac{\partial}{\partial x^j}X^i(x,\overline{u}^h) + \sum_{k=1}^{N}\frac{\partial X^i}{\partial y^k}(x,\overline{u}^h)(-\sum_{j=1}^{n} p_M^{ij}\frac{\partial(\overline{u}^h)^k}{\partial x^j} + p_M^{i\,n+k})].$$

Now noting that each vector $\nu_k := {}^t(-D(\overline{u}^h)^k, \varepsilon_k)$ is normal to M $(k=1,2,\dots,N)$, where $\{\varepsilon_1,\dots\varepsilon_N\}$ is the standard basis of $\boldsymbol{R}^N$, we find that the second term of (4.7) vanishes and by integration by parts

$$(4.8)\qquad (\delta V_t^h)(X) = -\sum_{i=1}^{n+N}\sum_{j=1}^{n}\int_\Omega (\frac{\partial}{\partial x^j}(p_M^{ij}|M(D\overline{u}^h)|)X^i(x,\overline{u}^h)dx.$$

Moreover by a straight forward calculus we have

$$p_M^{ij} = \frac{1}{|M(D\overline{u}^h)|^2}\sum_{|\alpha|+|\beta|=N}\sum_{|\gamma|+|\delta|=N} C^{ij}_{\alpha\beta\gamma\delta}M_\alpha^{\overline{\beta}}(D\overline{u}^h)M_\gamma^{\overline{\delta}}(D\overline{u}^h),$$

where $C^{ij}_{\alpha\beta\gamma\delta}$'s are constants. We have by (1.1) and Proposition 3.1 that, changing $q_0 > 1$ if necessary, $\{\| D_s M(D\overline{u}^h)\|_{L^{q_0}((0,T)\times\Omega')}\}$ is uniformly bounded with respect to h. Hence, for $Y \in C_0^1((0,\infty)\times U;\boldsymbol{R}^{n+N})$ with spt $Y \subset (0,T)\times W$, we have

$$|\int_0^\infty (\delta V_t^h)(Y)dt| \le C\sum_{s=1}^{n}\| D_s M(D\overline{u}^h)\|_{L^{q_0}((0,T)\times\Omega')}\sup|Y| \le C_{T,W}\sup|Y|,$$

where $C_{T,W}$ is a constant independent of h. This inequality implies that the linear functional

$$C_0^1((0,\infty)\times U;\boldsymbol{R}^{n+N}) \ni Y \longmapsto \int_0^\infty (\delta V_t^h)(Y)dt \in \boldsymbol{R}$$

has a unique extension to a functional L_h on $C_0^0((0,\infty)\times U;\boldsymbol{R}^{n+N})$ and that $|L_hY| \le C_{W,T}\sup|Y|$. By the Banach-Alaoglu theorem and the Banach-Steinhaus theorem there exists a subsequence (still denoted by $\{L_h\}$) and a functional L such that L_hY converges to LY for each $Y \in C_0^0((0,\infty)\times U;\boldsymbol{R}^{n+N})$ and for Y with spt $Y \subset (0,T)\times W$

$$(4.9)\qquad |LY| \le C_{T,W}\sup|Y|.$$

By the Riesz representation theorem there are a Radon measure ν on $(0,\infty)\times U$ and a ν measurable $\boldsymbol{R}^{n+N}$ valued function v with $|v|=1$, ν-a.e., such that

$$LY = \int_{(0,\infty)\times U} v\cdot Y d\nu.$$

Inequality (4.9) implies

$$\nu((0,T)\times W) = \sup\{\int_{(0,T)\times W} v\cdot Yd\nu; Y \in C_0^0((0,T)\times W;\boldsymbol{R}^{n+N}),\ |Y|\le 1\} \le C_{T,W}.$$

We define $\rho(A) = \nu(A\times W)$ for a Borel set $A \subset (0,T)$. It is well-known that for ρ-a.e. $t \in (0,T)$ there exists a probability Radon measure m_t on W such that $d\nu = dm_t(z)d\rho(t)$ (refer to [3, Chapter 1, Theorem 10]). Then for $Y \in C_0^0((0,T)\times W;\boldsymbol{R}^{n+N})$

$$(4.10)\qquad LY = \int_0^T\int_W v\cdot Ydm_td\rho.$$

Let X be a vector field in $C_0^1(U; \boldsymbol{R}^{n+N})$. It follows from (4.5) that $\mathrm{div}_S X(z)$ is continuous with respect to $(z, S) \in U \times G$. Noting (4.6), we have by Proposition 3.4 that for each $T > 0$ and $\psi \in L^{q_0'}(0,T)$

$$\lim_{h\to 0} \int_0^T \psi(t)\delta V_t^h(X)dt = \int_0^T \psi(t)\delta V_t(X)dt. \tag{4.11}$$

Thus, since $L_h(\psi X) = \displaystyle\int_0^\infty \psi(t)\delta V_t^h(X)dt$ and $\displaystyle\lim_{h\to 0} L_h Y = LY$, we have for $\psi \in C_0^0(0,\infty)$ and $X \in C_0^1(U; \boldsymbol{R}^{n+N})$

$$L(\psi X) = \int_0^\infty \psi(t)\delta V_t(X)dt. \tag{4.12}$$

For X with spt $X \subset W$ and for ψ with spt $\psi \subset (0,T)$ we have by (4.10) and (4.12) that

$$\int_0^\infty \psi(t)\delta V_t(X)dt = \int_0^T \psi(t) \int_W v\cdot X dm_t d\rho. \tag{4.13}$$

On the other hand, since $\{\| D_s M(D\overline{u}^h) \|_{L^{q_0}((0,T)\times\Omega')}\}$ is uniformly bounded with respect to h, we obtain that $\{D_j(p_M^{ij}|M(D\overline{u}^h)|)\}$ is uniformly bounded in $L^{q_0}((0,T)\times\Omega')$ with respect to h. Thus, it follows from (4.8) and (4.11) that there exists a constant $C'_{T,W}$ such that for each $\psi \in L^{q_0'}(0,T)$

$$\left|\int_0^T \psi(t)\delta V_t(X)dt\right| \le C'_{T,W} \,\|\psi\|_{L^{q_0'}(0,T)} \sup|X|. \tag{4.14}$$

Note that $m_t(W) = \sup\{\int_W v\cdot X dm_t; X \in C_0^0(W; \boldsymbol{R}^{n+N}),\ |X| \le 1\}$. Since m_t is a probability measure,

$$\sup\{\int_W v\cdot X dm_t; X \in C_0^0(W; \boldsymbol{R}^{n+N}),\ |X|\le 1\} = 1. \tag{4.15}$$

Combining (4.13), (4.14), and (4.15), we have

$$\left|\int_0^T \psi(t)d\rho(t)\right| \le C'_{T,W} \,\|\psi\|_{L^{q_0'}(0,T)}\,.$$

Thus the functional $\psi \mapsto \displaystyle\int_0^T \psi(t)d\rho(t)$ is bounded in $L^{q_0'}(0,T)$. Then there exists a function $\tilde\rho \in L^{q_0}(0,T)$ such that $\displaystyle\int_0^T \psi(t)d\rho(t) = \int_0^T \psi(t)\tilde\rho(t)dt$ for any $\psi \in L^{q_0'}(0,T)$. Putting $\tilde m_t = \tilde\rho(t)m_t$, we have by (4.13) that

$$\int_0^T \psi(t)\delta V_t(X)dt = \int_0^T \psi(t)\int_W v\cdot X d\tilde m_t dt.$$

Thus for $\mathcal{L}^1$-a.e. $t \in (0,T)$

$$\delta V_t(X) = \int_W v\cdot X d\tilde m_t.$$

Since T is arbitrary, we have the conclusion. Q.E.D.

Proof of Lemma 4.3. Let us fix a point z_0 in U and a positive number ρ and let $\{\phi_j\}$ be a sequence of functions in $C_0^0(U)$ such that $\phi_j \ge 0$, $\phi_j \equiv 1$ on $B_\rho(z_0)$, where $B_\rho(z_0)$ denotes the open ball with center at z_0 and radius ρ, and spt $\phi_j \subset B_{(1+\frac{1}{j})\rho}(z_0)$. Then

$$(4.16)\qquad (\mathcal{H}^n \llcorner G_{u(t,\cdot)})(B_\rho(z_0)) \leq \int_U \phi_j(z) d(\mathcal{H}^n \llcorner G_{u(t,\cdot)}).$$

Note that

$$(4.17)\qquad \int_U \phi_j(z) d(\mathcal{H}^n \llcorner G_{u(t,\cdot)}) = \int_\Omega \phi_j(x, u(t,x)) |M(Du(t,x))| dx$$
$$= \sup\{\int_\Omega \phi_j(x,u(t,x)) \sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x) M_{\bar{\alpha}}^{\beta}(Du(t,x))) dx; (f_{\alpha\beta}) \in \mathcal{C}\},$$

where $\mathcal{C}$ is as in Lemma 4.1. Let T be any positive number. Suppose that $\psi \in L^{q_0'}(0,T)$ is nonnegative. Then, since $|\sum_{|\alpha|+|\beta|=n} f_{\alpha\beta}(x)\xi^{\alpha\beta}(S)| \leq 1$ for $(f_{\alpha\beta})_{|\alpha|+|\beta|=n} \in \mathcal{C}$, we have by Lemma 4.1

$$(4.18)\qquad \int_0^T \psi(t) \int_U \phi_j(z) d(\mathcal{H}^n \llcorner G_{u(t,\cdot)}) dt \leq \int_0^T \psi(t) \int_U \phi_j(z) d\mu_{V_t} dt.$$

It follows from (4.16) and (4.18) that $(\mathcal{H}^n \llcorner G_{u(t,\cdot)})(B_\rho(z_0)) \leq \int_U \phi_j(z) d\mu_{V_t}$ for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$. By making $j \to \infty$ we obtain

$$(4.19)\qquad (\mathcal{H}^n \llcorner G_{u(t,\cdot)})(B_\rho(z_0)) \leq \mu_{V_t}(\overline{B_\rho(z_0)})$$

for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$.

Note that the null set $\{t \in (0,\infty); \text{(4.19) does not hold at } t\}$ depends on z_0 and ρ. Let Z and Q be countable dense subsets in U and in $(0,\rho_0)$ for some $\rho_0 > 0$, respectively. Then there exists a null set $N_1 \subset (0,\infty)$ such that, for each $t \in (0,\infty) \setminus N_1$, (4.19) holds whenever $z_0 \in Z$ and $\rho \in Q$. Since Z and Q are dense, we have

$$(4.20)\qquad (\mathcal{H}^n \llcorner G_{u(t,\cdot)})(B_\rho(z)) \leq \mu_{V_t}(\overline{B_\rho(z)})$$

for each $z \in U$, for each $\rho \in (0,\rho_0)$, and for each $t \in (0,\infty) \setminus N_1$. By Proposition 1.1 5), Proposition 2.1 1), and Proposition 3.3, $G_{u(t,\cdot)}$ is n-rectifiable for $\mathcal{L}^1$-a.e. t. Then we may suppose that, for each $t \in (0,\infty) \setminus N_1$, $G_{u(t,\cdot)}$ is n-rectifiable. For such a t, for $\mathcal{H}^n \llcorner G_{u(t,\cdot)}$-a.e. $z \in U$,

$$\lim_{\rho \to 0} \frac{(\mathcal{H}^n \llcorner G_{u(t,\cdot)})(B_\rho(z))}{\omega_n \rho^n} = 1,$$

where ω_n is the volume of the n-dimensional unit ball. Thus we have $\Theta_*^n(\mu_{V_t}, z) \geq 1$ for $\mathcal{H}^n \llcorner G_{u(t,\cdot)}$-a.e. $z \in U$, for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$. Step 1 implies that this holds for μ_{V_t}-a.e. $z \in U$, for $\mathcal{L}^1$-a.e. t.

It follows from Lemma 40.5 of [12] that Lemma 4.2 implies the existence of the density $\Theta^n(\mu_{V_t}, z)$ at μ_{V_t}-a.e. $z \in U$, for $\mathcal{L}^1$-a.e. t. Thus the conclusion holds. Q.E.D.

Finally we show Step 3, the end of the proof of which is at the same time the end of the proof of Theorem 1.2.

Proof of Step 3. By the definition of an n-rectifiable varifold we have, for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$,

$$(4.21)\qquad \int_{U\times G} \phi(z) \xi^{\bar{0}0}(S) dV_t(z,S) = \int_U \phi(z) \xi^{\bar{0}0}(T_z G_{u(t,\cdot)}) \theta_t(z) d(\mathcal{H}^n \llcorner G_{u(t,\cdot)}).$$

The right hand side of (4.21) coincides with $\int_\Omega \phi(x,u(t,x))\theta_t(x,u(t,x))dx$ by (2.3) and (2.4). On the other hand we have by (4.3) that for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$ the left hand side of (4.21) coincides with $\int_\Omega \phi(x,u(t,x))dx$. Then the conclusion follows. Q.E.D.

参考文献

[1] W. K. Allard, *On the first variation of a varifold*, Annals of Math. **95** (1972), 417 - 491.

[2] B. Dacorogna, Direct Method in the Calculus of Variations, Springer-Verlag, 1988.

[3] L. C. Evans, Weak convergence Methods for Nonlinear Partial Differential Equations, CBMS, Amer. Math. Soc. No. 74, 1990.

[4] D. Fujiwara and S. Takakuwa, *A varifold solution to the nonlinear equation of motion of a vibrating membrane*, Kodai Math. J. **9** (1986), 84-116; correction, ibid. **14** (1991), 310-311.

[5] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems, Annal. Math. Studies, **105**, Princeton University Press.

[6] M. Giaquinta, G. Modica and J. Souček, Cartesian Currents in the Calculus of Variations I, II, Springer, 1997.

[7] K. Kikuchi, *Constructing weak solutions in a direct variational method and an application of varifold theory*, J. Differential Equations **150** (1998), 1-23.

[8] K. Kikuchi, *Constructing gradient flows for quasiconvex functionals*, preprint.

[9] N. Kikuchi, *An approach to the construction of Morse flows for variational functionals*, in "Nematics Mathematical and Physical Aspects", ed.: J. M. Coron, J. M. Ghidaglia and F. Hélein, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **332**, Kluwer Acad. Publ., Dordrecht-Boston-London, 1991, 195-199.

[10] 長澤壯之, 離散的勾配流と発展方程式, 第16回発展方程式若手セミナー報告集, 1994, 1-20.

[11] C. B. Morrey, *Quasiconvexity and the lower semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25-53.

[12] L. Simon, Lectures on Geometric Measure Theory, Proceeding of the Centre for Mathematical Analysis, Australian National University, Canberra Vol.3, 1983.